Part 1: Hyperbolic PDEs and Ideal Magnetohydrodynamics (MHD)

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Conservation laws

Many problems in natural sciences and engineering are modelled following a simple principle: **The time rate of change of a quantity of interest u in a fixed volume is equal to the flux of u across the boundary of this volume.**

- Mathematical translation
  - Consider a domain $\Omega \subset \mathbb{R}^d$ and the quantity of interest u defined for all points $x \in \Omega$
  - This simple principle gives the integral form (or natural form) of the conservation law

$$\frac{d}{dt} \int_{\Omega} u \, dx = - \oint_{\partial\Omega} \vec{f} \cdot \vec{n} \, ds \tag{1}$$

where the flux $\vec{f} = \vec{f}(u, x, t)$ and the normal vector $\vec{n}$ points outward of the domain Ω

Conservation laws

- Examples for quantities of interest that are conserved
  - Mass, $u = \varrho(x,t)$
  - Momentum, $u = \varrho(x,t)\,\vec{v}(x,t)$
  - Energy, $u = E(x,t)$
- The form and structure of the fluxes $\vec{f}$ contain the physical and mathematical considerations and are specific to the problem
  - E.g. the mass flux is given by $\vec{f} = \varrho(x,t)\vec{v}(x,t)$
- Famous examples are the Euler equations in gas dynamics and the MHD equations in plasma physics

Partial differential equation

- If we assume that $\vec{f}$ is differentiable (smooth!) in space, we can apply Gauss' law to the surface integral and get

$$\frac{d}{dt}\int_{\Omega} u\,dx = -\int_{\Omega} \vec{\nabla}\cdot\vec{f}\,dx \tag{2}$$

- The conservation law holds for all domains Ω, thus we get the PDE form of the conservation law

$$u_t + \vec{\nabla}\cdot\vec{f} = 0 \tag{3}$$

Hyperbolic PDE

- It is possible to reformulate the PDE once more, by using the chain rule for the flux divergence

$$u_t + \vec{a}(u) \cdot \vec{\nabla}\, u = 0 \tag{4}$$

where $\vec{a}(u) := \frac{\partial \vec{f}}{\partial u}$ is the flux Jacobian
- Note that $\vec{a}(u)$ has the physical units of a velocity
- In case of a system, where u is a vector, $\vec{a}(u)$ are d matrices
- **We see that the assumption on differentiable fluxes $\vec{f}$ is not valid in general (shocks!). This is why the integral form (1) is the natural form, as no assumption on the smoothness is necessary.**

Hyperbolic PDE

- It is possible to mathematically classify the PDE by looking at the eigenvalues of the flux Jacobian matrices
  - If all eigenvalues are real numbers, the PDE is characterised as **hyperbolic**
  - Hyperbolic PDEs typically describe wave propagation phenomena
  - The eigenvalues are speeds of characteristic waves (e.g. sound speed)
  - Many conservation laws are hyperbolic PDEs (e.g. Euler, ideal MHD)

Linear Transport Equation

- The simplest example of a hyperbolic PDE/conservation law is the linear advection equation

$$\frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial x} = 0 \tag{5}$$

  where $\lambda > 0$ is a real constant

- Given an initial solution $u_0(x)$ at time $t = 0$, it is easy to see that the exact solution is

$$u(x,t) = u_0(x - \lambda\, t) \tag{6}$$

- From the structure of the PDE and its solution, two properties immediately follow

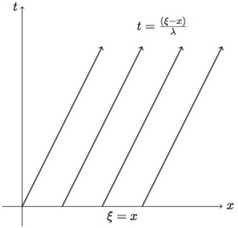
Linear Transport Equation

- Property 1: Conservation of the solution energy u^2
  - Multiply the advection equation by u and integrate over the domain
    $$\int_\Omega \frac{\partial u}{\partial t}\, u + \lambda \frac{\partial u}{\partial x}\, u \, dx = 0 \tag{7}$$
  - Use chain rule
    $$\int_\Omega \frac{1}{2} \frac{\partial u^2}{\partial t} + \lambda \frac{1}{2} \frac{\partial u^2}{\partial x} \, dx = 0 \tag{8}$$

    to see that u^2 is a conserved quantity with the flux $f(u) = \lambda\, u^2$
- Note, this property will play a crucial role in Part 2 about the numerical methods

Linear Transport Equation

- Property 2: Characteristics
  - The exact solution $u_0(x - \lambda\, t)$ shows that information $(u_0(x))$ is transported with the velocity $\lambda > 0$ along the curves $x(t) = \xi - \lambda\, t$
  - The curves $x(t)$ are straight lines and are called **characteristics**
  - Along the characteristics the solution is constant



- The slopes of the characteristics are $x'(t) = \lambda = const$

Non-linear case

- The simplest non-linear case is the Burgers' equation

$$
\begin{aligned}
u_t + u\, u_x &= 0 \\
u_t + (u^2/2)_x &= 0
\end{aligned}
\tag{9}
$$

  where the wave speed λ is no longer constant as $\lambda = u$

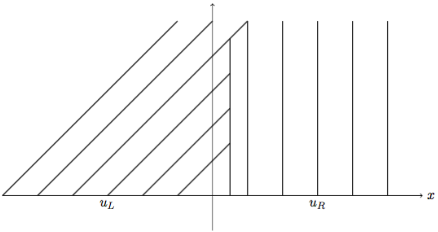
- It is still possible to formally define the characteristics
  - We construct curves with $x'(t) = \lambda = u$
  - It still holds that the solution is constant along the characteristics, and thus the curve is a straight line as well

Non-linear case

- Consider a special example, the so called **Riemann problem**

$$u_t + u\,u_x = 0, \quad u_0(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x > 0 \end{cases} \tag{10}$$

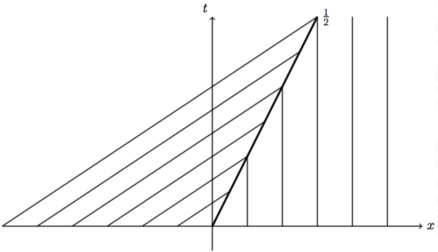
- The 'left' state $u_L = 1$ and the 'right' state $u_R = 0$ are constant, thus, the characteristics are parallel again FIX ME!!!!



- **The solution is constant along characteristics...what happens when they intersect?**

Non-linear case

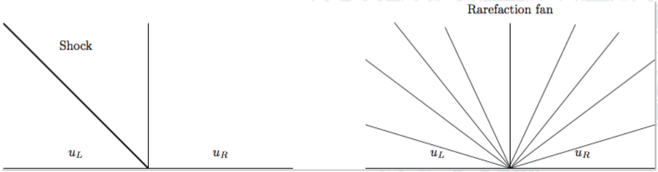
- This issue happens when the slope of the characteristics depends on the solution (non-linear case!)
- The characteristics say that the solution has two different values at the same point (discontinuity!)
- In physics, this is called a shock



- Note that the characteristics can also intersect for smooth initial conditions $u_0(x)$

Riemann problem

- The Riemann problem can be solved analytically
- Depending on the relation of u_L and u_R, different solutions result



- The shock speed (slope) s is computed with the Rankine Hugoniot condition $s = (f_R - f_L)/(u_R - u_L)$
- Note that for system of conservation laws, such as the ideal MHD, the Riemann problem is difficult to solve

Ideal MHD Conserved Quantities

- The ideal MHD equations are used to model the evolution of plasmas
- They are built from the conservation of eight quantities
  - Mass
  - Momenta
  - Total energy
  - Magnetic fields
- These eight quantities are collected into a vector

$$\vec{u} = (\varrho, \varrho\vec{v}, E, \vec{B})^T \tag{11}$$

where $\vec{v} = (u, v, w)^T$ and $\vec{B} = (B_1, B_2, B_3)^T$

Ideal MHD Fluxes

- The fluxes are built from the mathematical modelling as was discussed for the mass flux
  - The momenta fluxes are built from Newton's 2nd law
  - Total energy conservation comes from the First Law of Thermodynamics
  - The magnetic field fluxes are built from the laws of Faraday, Ohm, and Ampere (collectively the Maxwell's equations)

Ideal MHD Equations

- The ideal MHD equations have the form

$$\frac{\partial}{\partial t}\begin{bmatrix}\varrho\\ \varrho\vec{v}\\ E\\ \vec{B}\end{bmatrix} + \vec{\nabla}\cdot\begin{bmatrix}\varrho\vec{v}\\ \varrho(\vec{v}\vec{v}^T) + \left(p + \frac{1}{2}\|\vec{B}\|^2\right)\mathcal{I} - \vec{B}\vec{B}^T\\ \vec{v}\left(E + p + \frac{1}{2}\|\vec{B}\|^2\right) - \vec{B}(\vec{v}\cdot\vec{B})\\ \vec{v}\vec{B}^T - \vec{B}\vec{v}^T\end{bmatrix} = \vec{0} \tag{12}$$

- Closure of the system is done under the ideal gas assumption

$$p = (\gamma - 1)\left(E - \frac{\varrho\|\vec{v}\|^2}{2} - \frac{\|\vec{B}\|^2}{2}\right) \tag{13}$$

Two-Dimensional Ideal MHD Equations

- For the two-dimensional case considered in this pre-workshop the explicit forms of the fluxes in the x and y directions are

$$\vec{f} = \begin{bmatrix} \varrho u \\ \varrho u^2 + p + \frac{1}{2}\|\vec{B}\|^2 - B_1^2 \\ \varrho uv - B_1 B_2 \\ \varrho uw - B_1 B_3 \\ u\left(E + p + \frac{1}{2}\|\vec{B}\|^2\right) - B_1(\vec{v}\cdot\vec{B}) \\ 0 \\ uB_2 - vB_1 \\ uB_3 - wB_1 \end{bmatrix}, \quad \vec{g} = \begin{bmatrix} \varrho v \\ \varrho uv - B_1 B_2 \\ \varrho v^2 + p + \frac{1}{2}\|\vec{B}\|^2 - B_2^2 \\ \varrho vw - B_2 B_3 \\ v\left(E + p + \frac{1}{2}\|\vec{B}\|^2\right) - B_2(\vec{v}\cdot\vec{B}) \\ vB_1 - uB_2 \\ 0 \\ vB_3 - wB_2 \end{bmatrix} \tag{14}$$

respectively

Equivalent form of Ideal MHD Equations

- The vector of conservative variables is $\vec{u}$, but another set of **primitive variables** is of interest

$$\vec{\omega} = (\varrho, \vec{v}, p, \vec{B})^T \tag{15}$$

- The primitive variables are useful to rewrite the ideal MHD equations into quasi-linear form, e.g. in 1D,

$$\frac{\partial \vec{\omega}}{\partial t} + A \frac{\partial \vec{\omega}}{\partial x} = 0 \tag{16}$$

  where $A = \frac{\partial \vec{f}}{\partial \vec{\omega}}$ is the flux Jacobian matrix

- This is done because the flux Jacobian matrix from the primitive variables is much simpler than the flux Jacobian from the conservative variables

Form of the Flux Jacobian

$$A = \begin{bmatrix} u & \varrho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & 0 & \frac{1}{\varrho} & 0 & \frac{B_2}{\varrho} & \frac{B_3}{\varrho} \\ 0 & 0 & u & 0 & 0 & 0 & -\frac{B_1}{\varrho} & 0 \\ 0 & 0 & 0 & u & 0 & 0 & 0 & -\frac{B_1}{\varrho} \\ 0 & \gamma p & 0 & 0 & u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_2 & -B_1 & 0 & 0 & 0 & u & 0 \\ 0 & B_3 & 0 & -B_1 & 0 & 0 & 0 & u \end{bmatrix} \quad (17)$$

- However, the matrices are **similar** having the same eigenvalues but different eigenvectors
- This is very convenient because the wave speeds of the system are of great interest

Eigenvalues of the Flux Jacobian

- The flux Jacobian has eight eigenvalues

$$\lambda_{\pm f} = u \pm c_f, \quad \lambda_{\pm s} = u \pm c_s, \quad \lambda_{\pm a} = u \pm c_a, \quad \lambda_E = u, \quad \lambda_D = 0 \tag{18}$$

  where c_f, c_s are the fast and slow magnetoacoustic wave speeds and c_a is the Alfvén wave speed. The entropy wave (E) moves with the fluid velocity and the divergence wave (D) is stationary

- The wave speeds are computed by

$$c_a^2 = \tilde{b}_1^2, \qquad c_{f,s}^2 = \frac{1}{2}(\tilde{a}^2 + \tilde{b}^2) \pm \frac{1}{2}\sqrt{(\tilde{a}^2 + \tilde{b}^2)^2 - 4\tilde{a}^2\tilde{b}_1^2} \tag{19}$$

  with the convenient notation

$$\vec{\tilde{b}} = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3)^T = \frac{\vec{B}}{\sqrt{\varrho}}, \quad \tilde{b}^2 = \tilde{b}_1^2 + \tilde{b}_2^2 + \tilde{b}_3^2, \quad \tilde{a}^2 = \frac{p\gamma}{\varrho} \tag{20}$$

- The eigenvalues are well ordered and $|\lambda_{\pm f}|$ will always be the largest

Eigenvalues in Other Spatial Direction

- The eigenvalues for the flux Jacobian in the y direction are very similar

$$\lambda_{\pm f} = v \pm c_f, \quad \lambda_{\pm s} = v \pm c_s, \quad \lambda_{\pm a} = v \pm c_a, \quad \lambda_E = v, \quad \lambda_D = 0 \tag{21}$$

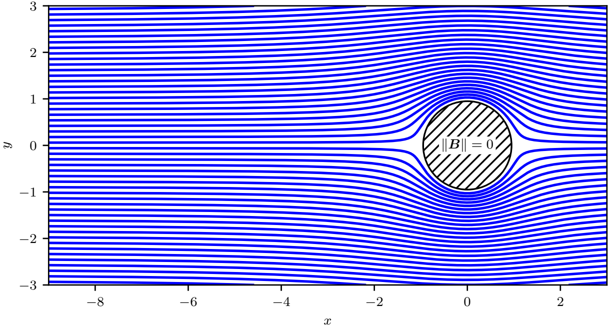
where the computation of c_a and $c_{f,s}$ uses b_2

Divergence-Free Flow

- An additional constraint not explicitly built into the ideal MHD equations is that the magnetic field remains divergence-free

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{22}$$

- This reflects the physical result that magnetic monopoles do not exist



Numerical Issues of $\vec{\nabla} \cdot \vec{B} = 0$

- On the continuous level the divergence-free constraint is assumed to always be satisfied due to Faraday's law
- However, for numerical methods it can be that the divergence-free constraint is violated (even if it is satisfied by the initial conditions)
- Partially this issue arises because the divergence wave of the system is stationary
- This has dramatic implications for the numerical modelling of plasmas as such errors can drive instabilities in the approximation
- Also, violation of the divergence-free constraint may create unphysical solutions as the shock speed of the approximate solution could no longer be correct

Discrete Recovery of $\vec{\nabla} \cdot \vec{B} = 0$

- Some of the methods available to address the numerical issues in the divergence-free condition are

  1. Projection methods explicitly remove the non-divergence-free parts of the magnetic field. Unfortunately, this is comes with a high computational cost because it requires the solution of a Poisson problem
  2. Constrained transport build the magnetic field components on a staggered grid and construct the fluxes in a certain way to ensure a divergence-free method
  3. Hyperbolic divergence cleaning introduces an additional variable proportional to $\vec{\nabla} \cdot \vec{B}$ governed by an advection equation. This new variable is then coupled into the induction equations to control the divergence errors.

Generalized Lagrange Multiplier (GLM) Method

- Select hyperbolic divergence cleaning because it is easy to implement and has a low numerical cost
- Introduce a new variable ψ governed by the advection equaiton

$$\frac{\partial \psi}{\partial t} + \vec{\nabla} \cdot \left(c_h^2 \vec{B} \right) = -\frac{c_h^2}{c_p^2} \psi \tag{23}$$

  where c_h is the propagation speed of the divergence error and $c_p \in (0, \infty)$

- A source term is also added for additional damping and control of the divergence error

GLM MHD Equations

- The ideal MHD equations augmented with GLM divergence cleaning take the form

$$\frac{\partial}{\partial t}\begin{bmatrix} \varrho \\ \varrho\vec{v} \\ E \\ \vec{B} \\ \psi \end{bmatrix} + \vec{\nabla}\cdot\begin{bmatrix} \varrho\vec{v} \\ \varrho(\vec{v}\vec{v}^T) + \left(p + \frac{1}{2}\|\vec{B}\|^2\right)\mathcal{I} - \vec{B}\vec{B}^T \\ \vec{v}\left(E + p + \frac{1}{2}\|\vec{B}\|^2\right) - \vec{B}(\vec{v}\cdot\vec{B}) \\ \vec{v}\vec{B}^T - \vec{B}\vec{v}^T + \psi\mathcal{I} \\ c_h^2\vec{B} \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{0} \\ 0 \\ \vec{0} \\ -\frac{c_h^2}{c_p^2}\psi \end{bmatrix} \tag{24}$$

- Note the system now contains nine equations and the vector of conserved variables is

$$\vec{u} = (\varrho, \varrho\vec{v}, E, \vec{B}, \psi)^T \tag{25}$$